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3-D LEAST SQUARES ADJUSTMENT PROGRAM FOR THE ADJUSTMENT AND ANALYSIS OF 3-D SURVEYING NETWORKS (PROGRAM METHODOLOGY AND USER GUIDE)

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by

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ABSTRACT

The development of the program 3D for the adjustment and analysis of 3-dimensional survey networks is presented in this paper. The program is seen as a powerful tool for mining surveying especially in the areas of control extension, control re-adjustment, and slope stability monitoring, but can be used for almost any 3-dimensional surveys in the mine.

The method of least squares is described as it applies to the reduction of redundant observations in survey networks. Mathematical models for the observation equations and their corresponding linearized forms incorporated into 3D are listed. Special techniques for saving computer memory and computation time are employed. These include sequential formation of the normal equations, elimination of the orientation unknowns, and efficient matrix inversion and storage algorithms.

Another section describes the program input and output files, program limiations, and program operation. The program file structure is very simple and makes the program very easy to use.

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PROGRAMME INFORMATIQUE 3D DE MISE AU POINT FONDÉ SUR LA MÉTHODE DES MOINDRES CARRÉS POUR METTRE AU POINT ET ANALYSER DES RÉSEAUX D'ARPENTAGE EN 3D (MARCHE À SUIVRE DU PROGRAMME ET GUIDE DE L'USAGER)

par

A.B. Kempe*

RÉSUMÉ

L'élaboration d'un programme informatique 3D pour mettre au point et analyser des réseaux d'arpentage tridimensionnels est présenté dans le présent rapport. Ce programme est perçu comme un outil important de l'arpentage minier, tout particulièrement dans les domaines du contrôle de l'extension et de la mise au point, et de la surveillance de la stabilité des pentes. Le programme peut servir aussi à presque toutes les études tridimensionnelles d'une mine.

La méthode des moindres carrés est décrite telle qu'elle s'applique à la réduction des observations superflues contenues dans les réseaux d'arpentage. Les modèles mathématiques des équations d'observation et leurs formes linéaires correspondantes incorporées en trois dimensions sont aussi présentées dans ce rapport. Des techniques spéciales conçues pour économiser la mémoire de l'ordinateur et le temps de calcul sont utilisées. Ces techniques comprennent la formation séquentielle des équations normales, l'élimination des inconnues d'orientation et l'inversion de matrice et le stockage d'algorithmes efficaces.

Enfin, une autre partie du rapport décrit les fichiers d'entrée et de sortie du programme, les restrictions et le mode d'exploitation du programme. L'organisation du fichier de ce programme est très simple et permet à l'usager de l'utiliser facilement.

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INTRODUCTION

The adjustment and analysis of survey networks is a very important procedure in an open pit mining operation. The precision and reliability of point coordinates determined through adjustment of survey observations has a direct influence on other mining operations. The effectiveness of blasting, for example, is directly influenced by accurate positioning of blast holes. Volume survey results are directly influenced by the accuracy of observing stations. The safety of men and equipment is directly influenced by the reliability of the mines slope stability monitoring program (predominately incorporating survey observations). Clearly precise and reliable point coordinate determination through appropriate mathematical processing is an important procedure which affects many aspects of the mining operation.

In this paper, the mathematical basis for a three dimensional least squares adjustment program "3D" is presented. Mathematical models for the observation equations and their corresponding linearized forms used in the program are listed. Space and time saving techniques incorporated into the program are discussed which enable the program to run efficiently on small computers. Tools used in the analysis of survey networks are also discussed. Finally, a description of the files used and created by the program is given, as well as program limits or restrictions, and program operation.

1.0 THE METHOD OF LEAST SQUARES

A well designed survey network is generally overdetermined. That is, more observations are taken than are necessary to uniquely define the values for the unknown coordinates. The advantages of having an overdetermined network include; (1) an increase in the accuracy of the results; (2) the ability to statistically analyze the observations for blunders; and (3) the ability of deleting bad observations without having to return to the field to re-observe.

Due to the stochastic nature of survey observations, the observations will not uniquely define the unknowns because of a lack of perfect fit to the math model. A least squares adjustment provides the means to solve a redundant, or over-determined system of linear equations. The least squares method is based on the condition that the weighted sum of the squares of the estimated residuals (corrections to the observables) is minimized. The least squares adjustment equations will be presented here without proof or derivation. Interested readers may refer to any of the many books and papers on the subject, eg. [Mikhail, 1976].

The least squares adjustment equations used in 3-D are given below. The linear parametric model is:

$$l = Ax + W \tag{1.1}$$

where x is the vector of unknown parameters,
l is the vector of observations,
W is the misclosure vector,
A is the design matrix.

The solution for the corrections to the unknowns $\hat{\delta}$ is:

$$\begin{aligned}\hat{\delta} &= -N^{-1}U \\ &= -(A^T C_l^{-1} A)^{-1} A^T C_l^{-1} W\end{aligned} \tag{1.2}$$

where N and U are called the "normal equations",
C_l is the covariance matrix of the observations,
$\hat{}$ indicates a least squares estimate.

The least squares estimate (adjusted value) of the unknowns is thus:

$$\hat{x} = x^{\circ} + \hat{\delta} \tag{1.3}$$

where x° is the approximate values of the unknowns x.

The covariance matrix of the estimated parameters is:

$$C_{\hat{x}} = N^{-1} = A^T C_l^{-1} A \ . \tag{1.4}$$

The adjusted observations $\hat{\mathbf{l}}$ are given by:

$$\hat{\mathbf{l}} = \mathbf{l} + \hat{\mathbf{r}} , \tag{1.5}$$

where $$\hat{\mathbf{r}} = \mathbf{A}\hat{\boldsymbol{\delta}} + \mathbf{W} \tag{1.6}$$

and is referred to as the vector of estimated residuals. The covariance matrix of $\hat{\mathbf{l}}$ is given by:

$$\begin{aligned} \mathbf{C}_{\hat{\mathbf{l}}} &= \mathbf{C}_{\mathbf{l}} - \mathbf{C}_{\hat{\mathbf{r}}} \\ &= \mathbf{A}\mathbf{C}_{\hat{\mathbf{x}}}\mathbf{A}^{T} \end{aligned} \tag{1.7}$$

where the covariance matrix of the estimated residuals ($\mathbf{C}_{\hat{\mathbf{r}}}$) is given by the expression:

$$\begin{aligned} \mathbf{C}_{\hat{\mathbf{r}}} &= \mathbf{C}_{\mathbf{l}} - \mathbf{C}_{\hat{\mathbf{l}}} \\ &= \mathbf{C}_{\mathbf{l}} - \mathbf{A}\mathbf{C}_{\hat{\mathbf{x}}}\mathbf{A}^{T}. \end{aligned} \tag{1.8}$$

A brief discussion of the computed values is now appropriate. The vector $\hat{\mathbf{x}}$ is the estimate of the unknown parameters and needs no further elaboration. The matrix $\mathbf{C}_{\hat{\mathbf{x}}}$ is the covariance matrix of $\hat{\mathbf{x}}$ and contains a complete statistical statement about the unknown parameters. It reflects both the accuracy of the observations and the effect of the geometry of the network on the accuracy of $\hat{\mathbf{x}}$ [Kempe, 1987].

The vector $\hat{\mathbf{l}}$ is the estimate of the true value of the observations. The observations, $\hat{\mathbf{l}}$, have random errors associated with them. The residual vector, $\hat{\mathbf{r}}$, is the estimate of these random errors. The analysis of the observations (Section 4.1) uses this vector to test the correctness of the vector of observations $\hat{\mathbf{l}}$. The associated covariance matrix $\mathbf{C}_{\hat{\mathbf{r}}}$ is also required for the analysis.

2.0 MATHEMATICAL MODELS

The relationship between the unknown parameters (x) and the survey observations made in the field (l), is through the formulation of the mathematical model. The parametric form of the mathematical model is given as:

$$l = f(x) \tag{2.1}$$

There are four math models used in 3D. They are all formulated on the conformal mapping plane and have the form $f(x) - l = 0$.
The spatial distance math model is:

$$[(N_j - N_i)^2 + (E_j - E_i)^2 + (H_j - H_i)^2]^{1/2} - SD_{ij} = 0, \tag{2.2}$$

where N is the unknown northing,
E is the unknown easting,
H is the unknown height, and
SD_{ij} is the observed slope distance between points i and j.

The zenith distance math model is:

$$\tan^{-1}\left[\frac{[(N_j - N_i)^2 + (E_j - E_i)^2]^{1/2}}{(H_j - H_i)}\right] - ZD_{ij} = 0 \tag{2.3}$$

where ZD_{ij} is the observed zenith distance between points i and j.

The differential height math model is:

$$H_j - H_i - DH_{ij} = 0 \tag{2.4}$$

where DH_{ij} is the observed height difference between points i and j.

Finally, the direction math model is:

$$\tan^{-1}\left[\frac{N_j - N_i}{E_j - E_i}\right] - \Omega - DR_{ij} \approx 0 \qquad (2.5)$$

where DR_{ij} is the observed direction from i to j, and
Ω is the orientation unknown.

The newly introduced parameter Ω simply transforms a direction to an azimuth. One orientation unknown is introduced for each round of directions. The elimination of this nuisance parameter is presented in Section 3.2

The mathematical models presented are non-linear while the least squares method illustrated earlier solves systems of linear equations and thus the math models must be linearized. This is done using a linear Taylor's series expansion and neglecting higher order terms as follows:

$$l = f(x) \approx f(x^\circ) + \left.\frac{df}{dx}\right|_{x=x^\circ} (x - x^\circ). \qquad (2.6)$$

This now represents the familiar expression from Equation 1.1 for the linear parametric model:

$$l = Ax + W \qquad (1.1)$$

with

$$A = \left.\frac{df}{dx}\right|_{x=x^\circ} \quad \text{the design matrix} \qquad (2.7)$$

and

$$W = f(x^\circ) - l^\circ \qquad (2.8)$$

where x° is the approximate values of the unknowns x, and
l° is the observed values of the observables l.

Noting that higher order terms of the Taylor's series expansion have been neglected, the solution must be iterated. This is accomplished by repeating the solution, replacing x° with the last $\hat{x}$ until the solution

has converged. The convergence criteria is simply when the corrections to the unknown parameters $\hat{x}$ become negligible.

The design matrix resulting from the linearization process is presented for each math model in the following discussion.
The design matrix for the linearized spatial distance math model is:

$$A_{ij} = \left[\frac{\partial f_{ij}}{\partial N_i} \frac{\partial f_{ij}}{\partial E_i} \frac{\partial f_{ij}}{\partial H_i} ; \frac{\partial f_{ij}}{\partial N_j} \frac{\partial f_{ij}}{\partial E_j} \frac{\partial f_{ij}}{\partial H_j} \right] \qquad (2.9)$$

where $\dfrac{\partial f_{ij}}{\partial N_i} = -\dfrac{(N_j - N_i)}{d^{\circ}_{ij}}$ $\qquad \dfrac{\partial f_{ij}}{\partial N_j} = -\dfrac{\partial f_{ij}}{\partial N_i}$

$\dfrac{\partial f_{ij}}{\partial E_i} = -\dfrac{(E_j - E_i)}{d^{\circ}_{ij}}$ $\qquad \dfrac{\partial f_{ij}}{\partial E_j} = -\dfrac{\partial f_{ij}}{\partial E_i}$

$\dfrac{\partial f_{ij}}{\partial H_i} = -\dfrac{(H_j - H_i)}{d^{\circ}_{ij}}$ $\qquad \dfrac{\partial f_{ij}}{\partial H_j} = -\dfrac{\partial f_{ij}}{\partial H_i}$

with

$$d^{\circ}_{ij} = [(N_j - N_i)^2 + (E_j - E_i)^2 + (H_j - H_i)^2]^{1/2}$$

The design matrix for the linearized zenith distance math model is:

$$A_{ij} = \left[\frac{\partial f_{ij}}{\partial N_i} \frac{\partial f_{ij}}{\partial E_i} \frac{\partial f_{ij}}{\partial H_i} ; \frac{\partial f_{ij}}{\partial N_j} \frac{\partial f_{ij}}{\partial E_j} \frac{\partial f_{ij}}{\partial H_j} \right] \qquad (2.10)$$

where $\dfrac{\partial f_{ij}}{\partial N_i} = -\dfrac{(N_j - N_i)(H_j - H_i)}{(d^{\circ}_{ij})^2 ((N_j - N_i)^2 + (E_j - E_i)^2)^{1/2}}$

$\dfrac{\partial f_{ij}}{\partial E_i} = -\dfrac{(E_j - E_i)(H_j - H_i)}{(d^{\circ}_{ij})^2 ((N_j - N_i)^2 + (E_j - E_i)^2)^{1/2}}$

$\dfrac{\partial f_{ij}}{\partial H_i} = \dfrac{((N_j - N_i)^2 + (E_j - E_i)^2)^{1/2}}{(d^{\circ}_{ij})^2}$

$\dfrac{\partial f_{ij}}{\partial N_j} = -\dfrac{\partial f_{ij}}{\partial N_i}$ $\qquad \dfrac{\partial f_{ij}}{\partial E_j} = -\dfrac{\partial f_{ij}}{\partial E_i}$ $\qquad \dfrac{\partial f_{ij}}{\partial H_j} = -\dfrac{\partial f_{ij}}{\partial H_i}$

with $\overset{\circ}{d}_{ij} = [(N_j - N_i)^2 + (E_j - E_i)^2 + (H_j - H_i)^2]^{1/2}$

The design matrix for the linearized differential height math model is simply:

$$A_{ij} = \left[\frac{\partial f_{ij}}{\partial H_i} \; ; \; \frac{\partial f_{ij}}{\partial H_j} \right] \qquad (2.11)$$

where $\frac{\partial f_{ij}}{\partial H_i} = -1 \qquad \frac{\partial f_{ij}}{\partial H_j} = 1$

Finally, the design matrix for the linearized direction math model is:

$$A_{ij} = \left[\frac{\partial f_{ij}}{\partial N_i} \; \frac{\partial f_{ij}}{\partial E_i} \; ; \; \frac{\partial f_{ij}}{\partial N_j} \; \frac{\partial f_{ij}}{\partial E_j} \right] \qquad (2.12)$$

where $\frac{\partial f_{ij}}{\partial N_i} = -\frac{(E_j - E_i)}{\overset{\circ}{d}_{ij}} \qquad \frac{\partial f_{ij}}{\partial N_j} = -\frac{\partial f_{ij}}{\partial N_i}$

$$\frac{\partial f_{ij}}{\partial E_i} = \frac{(N_j - N_i)}{\overset{\circ}{d}_{ij}} \qquad \frac{\partial f_{ij}}{\partial E_j} = -\frac{\partial f_{ij}}{\partial E_i}$$

and $\overset{\circ}{d}_{ij} = (N_j - N_i)^2 + (E_j - E_i)^2$

3.0 SPACE AND TIME SAVING TECHNIQUES

A number of techniques have been incorporated into 3D which enable a large least squares problem to fit onto a small computer. The techniques described in the following illustrate methods of efficient storage and inversion of matrices.

3.1 SEQUENTIAL FORMATION OF THE NORMAL EQUATIONS

The design matrix A is the largest matrix in the set of least squares equations, having one row per observation and one column per unknown. The matrix is also very sparse. At most, only six elements of

a row are non-zero. In this section, a simply technique is presented for the total elimination of this matrix.

If the A matrix is partitioned into rows ($[a_i]$ denotes the ith row of A),

$$A = \begin{bmatrix} [a_1] \\ [a_2] \\ \vdots \\ [a_n] \end{bmatrix} \tag{3.1}$$

the diagonal C_l matrix into its individual elements

$$C = \begin{bmatrix} C_{l_1} & 0 & 0 \\ 0 & C_{l_2} & \ddots \; 0 \\ 0 & 0 & C_{l_n} \end{bmatrix} \tag{3.2}$$

and the misclosure vector W into its individual elements

$$W = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{bmatrix} \tag{3.3}$$

and substitute these into the equations for N and U (Equation 1.2) we get

$$N = [a_1]^T C_{l_1}^{-1} [a_1] + [a_2]^T C_{l_2}^{-1} [a_2] + \cdots + [a_n]^T C_{l_n}^{-1} [a_n] \tag{3.4}$$

and

$$U = [a_1]^T C_{l_1}^{-1} W_1 + [a_2]^T C_{l_2}^{-1} W_2 + \cdots + [a_n]^T C_{l_n}^{-1} W_n. \tag{3.5}$$

From these equations it is apparent that the normal equations N and U can be formed sequentially (by summation) one observation at a time and that no A matrix or W vector are needed. An additional saving in space

and execution time is obtained if only the non-zero elements of each rov of the A matrix are used in the computations.

This technique, however, is valid only if the C_l matrix is a diagonal matrix. The elimination of the orientation unknovns (Section 3.2) associated with a round of directions creates a block diagonal C_l matrix. In this case, the A matrix is partitioned into blocks of rovs corresponding to all the observations within the round and similarly to the block diagonal structure of the C_l matrix.

The equation for $\hat{r}$ given in Section 1.0 is:

$$\hat{r} = A\hat{\delta} + W \; ; \qquad (1.6)$$

however using the technique just illustrated, there is no A matrix or W vector available for the computation of $\hat{r}$. This difficulty is overcome by noting that:

$$f(\hat{x}) - \hat{l} = 0 \qquad (3.6)$$

and

$$\hat{l} = l + \hat{r}. \qquad (3.7)$$

Substituting Equation 3.7 into Equation 3.6 gives:

$$\hat{r} = f(\hat{x}) - l. \qquad (3.8)$$

Note that Equation 3.8 is exact while Equation 1.6 is actually a linear approximation.

3.2 ELIMINATION OF ORIENTATION UNKNOVNS

Recall that the mathematical model for direction observations (Eqn. 2.5) introduced a nev unknovn, Ω, for each round of directions. There are three coordinate unknovns and, typically, one orientation unknovn per occupied station in a survey network. The time required to compute the inverse, or Cholesky root, of a matrix is proportional to u^3.

Therefore the elimination of the orientation unknowns will reduce the execution time by a factor of $3^3 / 4^3$ or more than 40 %.

The orientation unknowns can be eliminated as follows [Steeves, 1974]. Partition the δ vector into δ_1 corresponding to the coordinate unknowns and δ_2 corresponding to the orientation unknowns. The A matrix must be similarly partitioned. This leads to the equation (from Eqn. 1.2):

$$[A_1 \; A_2]^T C_1^{-1} [A_1 \; A_2] \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} + [A_1 \; A_2]^T C_1^{-1} W = 0 \qquad (3.9)$$

which, when expanded, becomes

$$\begin{bmatrix} A_1^T C_1^{-1} A_1 & A_1^T C_1^{-1} A_2 \\ A_2^T C_1^{-1} A_1 & A_2^T C_1^{-1} A_2 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} + \begin{bmatrix} A_1^T C_1^{-1} W \\ A_2^T C_1^{-1} W \end{bmatrix} = 0. \qquad (3.10)$$

Applying the identity [Fadeev and Fadeeva, 1963] that, if

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} U \\ V \end{bmatrix} = 0, \qquad (3.11)$$

then

$$[\, D - CA^{-1}B \,]\, Y + V - CA^{-1}U = 0, \qquad (3.12)$$

to Equation 3.10 to obtain

$$\delta_1 = - [A_1^T Q A_1]^{-1} [A_1^T Q W], \qquad (3.13)$$

where $$Q = C_1^{-1} - C_1^{-1} A_2 [A_2^T C_1^{-1} A_2]^{-1} A_2^T C_1^{-1}. \qquad (3.14)$$

Equation 3.13 is identical in form to the least squares equations given previously with the new matrix Q replacing C_1^{-1}. Noting that A_2 consists entirely of partial columns of negative ones and C_1 is a diagonal matrix, the computation of Q is simply:

$$q_{i,j} = p_{i,j} - p_{i,i} * p_{j,j} * z \qquad (3.15)$$

where $q_{i,j}$ is the (i,j) element of Q,

$$z = \left[\sum_{k=1}^{n} p_{k,k} \right]^{-1} \qquad (3.16)$$

and $p_{i,j}$ is the (i,j) element of C_1^{-1}.

The computation of $\hat{\mathbf{r}}$ for directions using Equation 3.8 requires the adjusted value of the orientation unknown. Steeves [1974] shows that it may be computed after the adjustment and consequently the residuals computed as follows:

1. Compute an initial approximation $\tilde{\mathbf{r}}$ from Equation 3.8 letting $\mathbf{Q} = 0$;
2. Multiply $\tilde{\mathbf{r}}$ by $-[p_{1,1} * z, p_{2,2} * z, \ldots, p_{nn} * z]$ to get Q;
3. Subtract Q from each element of $\tilde{\mathbf{r}}$ to get $\hat{\mathbf{r}}$.

3.3 MATRIX INVERSION

All the matrices requiring inversion when using the method of least squares as described in Section 1.0 are symmetric and positive definite. Special algorithms have been developed that take advantage of the symmetry of matrices to economize on both the storage space and computational effort required for their inversion. Cholesky factorization is one such algorithm that is used in 3D, and is discussed in the remainder of this section.

A positive definite matrix, Q, can be factored into the product of the transpose of an upper triangular matrix and itself [Burden et al, 1978], namely

$$\mathbf{R}^T\mathbf{R} = \mathbf{Q}. \qquad (3.17)$$

Substituting Equation 3.17 into the least squares solution gives:

$$\begin{aligned} \hat{\boldsymbol{\delta}} &= -\mathbf{N}^{-1}\mathbf{U} \\ &= -(\mathbf{R}^T\mathbf{R})^{-1}\mathbf{U} = -\mathbf{R}^{-1}\mathbf{R}^{-T}\mathbf{U}. \end{aligned} \qquad (3.18)$$

Since the matrix R is triangular, the equations

$$d = R^{-T}U \tag{3.19}$$

and

$$\hat{\delta} = -R^{-1}d \tag{3.20}$$

can be solved directly with no explicit inverse computations. The algorithms are as follows:

For k = 1 to n

$$d_k = \left[u_k - \sum_{i=1}^{k-1} r_{i,k} * u_i \right] / r_{k,k}$$

End k

and

For k = n to 1 by -1

$$\hat{\delta}_k = -d_k / r_{k,k}$$

For i = 1 to k-1

$$\hat{\delta}_i = \hat{\delta}_i - \hat{\delta}_k * r_{i,k}$$

End i

End k.

These are referred to, respectively, as the forward and backward solutions. The inverse of the normal equations, when required, is computed from:

$$N^{-1} = (R^T R)^{-1} = R^{-1}R)^T. \tag{3.21}$$

Examination of these algorithms shows that it is possible to overwrite N with R, R with N^{-1}, U with d, and d with $\hat{\delta}$. The result is that no matrices other than N and U are required for the solution. Also, only the upper triangular elements of N are stored in a vector, therby saving just under 50% of the space required to store the full N matrix [Dongarra et al, 1979].

It is also apparent that it is not necessary to compute N^{-1} on every iteration. Only the Cholesky root, R, and the forward and backward solutions are needed. This aspect results in a saving in computation time of approximately 50%.

4.0 ANALYSIS OF RESULTS

The results of an adjustment must be analysed to ensure that they are correct and that the accuracy specifications have been met. The analysis of the results is at least as important as observing and adjusting of the data.

The following section presents statistical tests used in the analysis of the observations. The tests presented will not give a clear yes-no answer to the hypothesis that the assumption being tested is valid. Instead, testing is done at some confidence level $1-\alpha$. If, for example, α equals 0.05, this corresponds to a confidence level of 95%. This allows us to say, for example, that we are 95% confident the observation is a blunder if it fails the test given in Section 4.1.1. Note that in this case, about 5% of the observations rejected are not blunders. The chance of rejecting good data is decreased by decreasing α but the chance of accepting bad data is increased. The choice of the confidence level to be used in statistical testing must be made carefully.

4.1 ASSESSMENT OF THE OBSERVATIONS

The following tests are presented without proof or derivation, interested readers may refer to [Vanicek and Krakiwsky, 1982] for a more detailed presentation. The purpose of the test, the computation of the test, and the interpretation of the test are presented.

4.1.1 DETECTION OF OUTLIERS (TEST OF THE RESIDUALS)

The purpose of this test is to detect blunders in the observations. This test requires that the distribution of the observations is known. Under the assumption that the observations have a normal probability density function (pdf), it can be shown that the least squares estimates of the residuals will also have a normal distribution. In general, however, each residual has a different variance and thus a different normal distribution. In order to transform the residuals into variables which all have the same distribution, we devide them by their standard deviations. This process is called "standardization" of the residuals. In the case of the variance factor (σ_o^2) known, the standardized residuals

$$\hat{r}_i^{\,*} = \hat{r}_i \,/\, \sigma_{\hat{r}_i} \qquad (4.1)$$

have a normal distribution with mean zero and standard deviation equal to unity. The standard deviation of the residuals, $\sigma_{\hat{r}_i}$, are the square roots of the diagonal elements of $C_{\hat{r}}$. Thus for an observation to be rejected, it must fall outside the confidence interval of [Vanicek and Krakivsky, 1982]

$$\left[-\xi_{n(0,1)_{1-\alpha/2}} < \frac{\hat{r}_i}{\sigma_{\hat{r}_i}} < \xi_{n(0,1)_{1-\alpha/2}} \right] \qquad (4.2)$$

In the above test we are examining each residual by itself, out of context of the other residuals. Since we are really concerned about the characteristics of the entire set of residuals we should base our confidence intervals on simultaneous (or "in context") probability statements, in which all of the standardized residuals are simultaneously within a specified interval with a certain probability. To convert to an in context test, simply replace α with α/n where n is the number of observations being tested in the context of each other. 3D uses in context testing.

An observation may be rejected for any of the following reasons:

1. The observation is indeed a blunder;
2. The observation is not from a normally distributed population; or
3. The wrong (too small) standard deviation was assigned to the observation.

4.1.2 TEST OF THE VARIANCE FACTOR

This test is used to verify that the scale of the C_l matrix was chosen correctly and is indeed σ_o^2. The test is:

$$\left[\frac{\nu \hat{\sigma}_o^2}{\xi \chi^2_{\nu,1-\alpha/2}} < \sigma_o^2 < \frac{\nu \hat{\sigma}_o^2}{\xi \chi^2_{\nu,\alpha/2}} \right] \qquad (4.3)$$

where σ_o^2 is the a priori variance factor,

$\hat{\sigma}_o^2$ is the estimated (a posteriori) variance factor

$$\hat{\sigma}_o^2 = \frac{\hat{r}^T C_l^{-1} \hat{r}}{\nu} \qquad (4.4)$$

ν is the degrees of freedom of the adjustment
ν = number of observation equations minus the number of unknown parameters.

The test may fail for any of the following reasons:

1. The apriori variance factor was incorrect;
2. The observations were not normally distributed;
3. There may be systematic or gross errors in the observations.

If it is believed that σ_o^2 was chosen incorrectly, simply rescale the C_l matrix by $\hat{\sigma}_o^2$ and re-adjust the network.

4.2 ASSESSMENT OF THE PARAMETERS

As mentioned earlier, the covariance matrix of the unknowns $C_{\hat{x}}$, fully describes the accuracy of the coordinates determined from a least squares adjustment. From the variance-covariance information within the $C_{\hat{x}}$ matrix, confidence regions in the form of ellipsoids are determined for the unknown stations which reflect the accuracy of the computed positions relative to the fixed stations. These are referred to as point confidence ellipsoids.

In addition, the $C_{\hat{x}}$ matrix enables the computation of relative confidence ellipsoids. These confidence regions are used to indicate the accuracy that one station has been determined to, relative to another station. Note that for an observation between a fixed and unknown station, the relative confidence ellipsoid is equal to the point confidence ellipsoid for the unknown station. Thus relative confidence ellipsoids yield new information only between unknown stations. The relative confidence ellipsoid is calculated using the covariance matrix of the coordinate differences computed from the adjusted coordinates. The covariance law is applied to the $C_{\hat{x}}$ matrix to yield:

$$C_{\Delta\hat{x}} = G C_{\hat{x}}^{*} G^{T} \qquad (4.4)$$

where

$$G = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

and

$C_{\hat{x}}^{*}$ is the 6 by 6 submatrix of $C_{\hat{x}}$ corresponding to the two stations of interest.

Interested readers may refer to [Vanicek and Krakiwsky, 1982] for a more detailed description of the computation of these confidence regions.

5.0 PROGRAM FILE DESCRIPTION, LIMITATIONS, AND OPERATION

5.1 INPUT DATA FILE

Program 3D requires only one data file (called 3dinput.dat). This data file contains all the station coordinates (actual for fixed stations, approximate for unknown) and all the survey observations. For simplicity, the station names and coordinates are stored in a particular format, while all the observations are stored in another format. Strict adherence to this format must be followed to ensure all significant figures are read from the file during execution. A sample file with the exact format is given below:

```
FX   F1  3812.107  1040.273 1022.141
UK    A  3478.916  1499.993 1201.933
EN   EN 00000.000 00000.000    0.000
DR   F1     D   23.      57.    58.7  2.00 1.00 2.00  1.442 1.450
SD   F1     D  676.0230  5.00  2.00 1.00 2.00 1.412 1.504 0.0
ZD   F1     D   69.      13.    31.4  2.00 1.00 2.00  1.457 1.472
DH   F1     H   42.718    2.00 3.125 0.0  0.0  0.0   0.0   0.0
EN   EN    EN   00.000    0.00 0.000 0.0  0.0  0.0   0.0   0.0
12345678901234567890123456789012345678901234567890123456789012345
  ↳ Column #
```

where **FX** denotes a fixed station;
UK denotes an unknown station;
EN denotes end of data type and end of file (**must be included**);
DR denotes a direction observation;
SD denotes a slope distance observation;
ZD denotes a zenith distance observation; and
DH denotes a differential height observation.

It must be remembered that the direction observations are processed by "round" and thus each round must be appear in the data file sequentially. The program has the ability to recognize individual rounds through examination of "TO" and "FROM" station names. Note also that the spelling of the "codes" for the stations and observations must be correct and upper-case for processing to occur correctly.

5.2 ECHOED-INPUT FILE

The program creates an "echo" of the input data file for checking purposes and is simply called "Input.dat". Below is a sample of the echoed data illustrating the units of the raw data file and what each column represents:

STATUS	STA	NORTHING (m)	EASTING (m)	ELEVATION (m)
FX	F1	3812.107	1040.273	1022.141
UK	A	3478.916	1499.993	1201.933

STATUS	FROM	TO	DIR (deg)	S.D. (sec)	I.C.E. (mm)	T.C.E. (mm)	HI (m)	HT (m)
DR	F1	D	23.9663	2.00	1.00	2.00	1.442	1.450

STATUS	FROM	TO	DIS (m)	A (mm)	B (ppm)	I.C.E. (mm)	T.C.E. (mm)	HI (m)	HT (m)
SD	F1	D	676.023	5.00	2.00	1.00	2.00	1.412	1.504

STATUS	FROM	TO	ZEN (deg)	S.D. (sec)	I.C.E. (mm)	T.C.E. (mm)	HI (m)	HT (m)
ZD	F1	D	69.2254	2.00	1.00	2.00	1.457	1.472

STATUS	FROM	TO	H.D. (m)	S.D. (m)	Level Dist (km)
DH	F1	H	42.718	2.00	3.125

where **FROM** is the occupied station;
TO is the observed station;
S.D. is the standard deviation of the observation, and should include the effects of pointing, reading and levelling errors;
I.C.E. is the instrument centering error;
T.C.E. is the target centering error;
HI is the instrument height;
HT is the target height;
A is the constant portion of the EDM accuracy specification;
B is the distance dependent portion of the EDM accuracy specification;
S.D. (for DH) is the constant component of the height difference standard deviation; and,
Lev Dist is the levelled distance between the two stations.

Note that the degree-minute-second for directions and zenith distances has been converted to its corresponding degree value simply for ease of representation.

5.3 OUTPUT FILE

The output file created by the program is simply called "Output.dat". A sample of the output is found in the appendix and the following discussion will refer to that output.

Firstly, the program prints the adjusted coordinates for the unknown stations. These are quite self-explanatory. Next is the internal reliability analysis. Shown is the residual, redundancy number, and minimally detectable blunder for each observation in the network. Examination of the results reveals that observations with low redundancy numbers (ie. weak geometric connection to the network) have correspondingly high minimally detectable blunders, while observations with high redundancy numbers (strong geometric connection to the network) are much more sensitive to observational blunders. For a more detailed discussion of these quantities, readers should refer to [Kempe, 1987]. Included for each observation is the 99% residual test for outliers (Section 4.1.1). If the observation passes the test nothing is printed, however, in the event of failure of the test the observation is flagged with the word "REJECT". Internal reliability measures provide the means for surveyors to detect blunders and either eliminate them (by deleting them from the data set, or re-observing, or assigning negligible weight in the adjustment), or simply recognize their existence and interpret the results accordingly.

Next is printed the results of the test of the variance factor (Section 4.1.2). Printed are the a priori and a posteriori variance factors. Also printed is the bounds of the confidence region corresponding to the test and a sentence indicating pass or failure of the test.

Finally, the output file contains the 99% point confidence analysis and relative confidence analysis. Again these quantities are self-explanatory and need no further elaboration.

5.4 PROGRAM LIMITATIONS (RESTRICTIONS)

Basically the only limiting factor for the size of a network to be adjusted with 3D is the FORTRAN requirement that a maximum size for an array be declared when a program is created. This is not entirely true though, as memory bounds especially on the PC version of the program limit the size of network as well.

The limits established in 3D are;

1. A maximum of 10 fixed stations and 30 unknown stations may be in the network at any one time. (20 unknown for PC)
2. A maximum of 500 observations may be in the network at any one time.
3. A maximum of 32 directions may be in a single round. (22 for PC version)
4. A maximum 4 characters for station names.
5. A maximum slope distance of 9999.999 meters can be observed.

To change these limits, simply change the values of the parameters located in the parameter statements. This is not recommended however.

5.5 PROGRAM OPERATION

Once the data file has been created under the proper name "3dinput.dat", simply type 3D and hit the carriage return. This executes the program and no other interaction is required. During execution, the current iteration of the solution will be printed on the screen until convergence has been reached. Once the program execution is complete the two output files may be examined.

CONCLUSIONS AND RECOMMENDATIONS

In this paper, the mathematical basis for a three dimensional least squares adjustment program "3D" was presented. The presentation is directed toward persons interested in program details. The presentation is not directed toward a detailed explanation of the results from the program and interested readers should consult the numerous books and

papers on the subject for more information on interpretation of the results.

The space and time saving techniques presented in Section 3.0 enable the program to run efficiently, even on small computers. The precision and reliability measures illustrated in Section 4.0 give accuracy estimates for the unknowns, enable blunder detection, and give insight into the controllability of observations in the survey network. Section 5.0 gives a description of the files used and created by the program, program limitations or restrictions, and a simple description of how to use the program.

Several extensions to the program could include graphics capabilities, ability to incorporate other observation types such as satellite range or phase difference measurements, simpler method of data input, the inclusion of sequential expressions to allow the rapid addition and deletion of observations to the network. Indeed many extensions to the program are possible, however, the existing program serves as an excellent starting point and presently incorporates many techniques and determines many quantities never previously utilized in a mining surveying environment.

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OUTPUT

ADJUSTED COORDINATES

STATUS	STA	NORTHING (m)	EASTING (m)	ELEVATION (m)
UK	A	3479.621	1501.710	1203.045
UK	B	3493.910	2316.399	1222.555
UK	C	3417.442	3003.509	1213.879
UK	D	3210.735	845.759	1261.896
UK	E	2822.921	1579.948	1289.400
UK	F	2814.178	2389.634	1249.564
UK	G	2817.432	3032.332	1275.530
UK	H	2118.678	1054.132	1064.864
UK	I	1536.531	2104.076	1087.918

INTERNAL RELIABILITY ANALYSIS

STATUS	FROM	TO	RESIDUAL (sec/mm)	RED No (%)	MIN DET BLUNDER (sec/mm)	99% RESIDUAL TEST FOR OUTLIERS
DR	F1	D	0.29	1	94.69	
DR	F1	H	0.18	8	23.71	
DR	F1	I	0.69	22	14.61	
DR	F1	E	-0.73	54	9.46	
DR	F1	A	-1.22	21	16.05	
DR	F1	B	0.52	63	8.75	
DR	F1	F2	0.13	67	8.36	
DR	F2	F1	0.44	86	7.39	
DR	F2	D	3.12	77	7.87	
DR	F2	A	-1.82	70	8.27	
DR	F2	E	-0.22	33	12.38	
DR	F2	I	-0.53	13	19.39	
DR	F2	F	-1.01	12	22.26	
DR	F2	B	-1.95	66	8.63	
DR	F2	F4	2.92	86	7.43	
DR	F2	G	-0.77	51	9.84	
DR	F2	C	0.22	71	8.23	
DR	F2	F3	-0.75	86	7.39	
DR	F3	F2	0.07	87	7.35	
DR	F3	A	-1.75	83	7.56	
DR	F3	B	0.16	69	8.27	
DR	F3	E	4.24	84	7.52	
DR	F3	H	-0.53	74	7.99	
DR	F3	F	-2.78	79	7.79	
DR	F3	C	-0.17	17	17.97	
DR	F3	I	0.92	46	10.11	
DR	F3	G	0.64	61	8.96	
DR	F3	F4	-0.83	87	7.35	
SD	F1	D	2.91	9	65.84	
SD	F1	H	-0.29	3	137.80	
SD	F1	E	4.53	46	29.93	
SD	F1	A	0.18	38	30.92	
SD	F2	A	0.36	54	28.24	
SD	F2	E	1.31	34	33.20	
SD	F2	F	1.46	34	32.36	
SD	F2	B	-0.56	29	37.32	
SD	F2	G	2.71	31	35.44	
SD	F2	C	6.05	53	28.30	
SD	F3	B	-3.87	24	41.11	
SD	F3	F	-0.59	57	27.97	
SD	F3	G	3.16	36	33.67	
SD	F3	C	5.97	43	29.32	
ZD	F1	D	0.50	2	52.94	
ZD	F1	H	-0.80	98	6.97	
ZD	F1	I	-1.22	99	6.90	
ZD	F1	E	0.80	71	8.27	
ZD	F1	A	-0.14	24	14.87	
ZD	F2	A	0.34	81	7.71	
ZD	F2	E	0.12	39	11.35	
ZD	F2	F	-0.48	16	18.63	
ZD	F2	B	-1.52	48	10.07	

ZD	F2	G	0.58	46	10.40
ZD	F2	C	2.69	81	7.70
ZD	F3	B	1.25	54	9.52
ZD	F3	F	1.64	89	7.35
ZD	F3	G	0.07	60	9.02
ZD	F3	I	0.65	99	6.89
ZD	F3	C	-0.62	22	15.52
DH	F1	H	4.89	52	16.73
DH	H	I	2.15	22	16.72
DH	I	F2	2.97	30	16.46

RESULTS OF THE TEST ON THE VARIANCE FACTOR

A PRIORI VARIANCE FACTOR = 1.000

ESTIMATED (A POSTERIORI) VARIANCE FACTOR = 0.814

GLOBAL TEST OF THE ESTIMATED VARIANCE FACTOR

THE TEST IS 0.469 < 1.000 < 1.679

AT 99% CONFIDENCE LEVEL THE TEST PASSES !

99 % POINT CONFIDENCE ANALYSIS

APRIORI VARIANCE FACTOR = 1.000

EXPANSION FACTORS = [2.576 (1D), 3.035 (2D), 3.368 (3D)].

STATION NAME	3 DIMENSIONAL ELLIPSOIDS AXES METERS	ORIENTATION TO RESPECTIVE PLANES DEGREES NE	EH	NH	2D PROJECTED CONFIDENCE ELLIPSES A METERS	B METERS	AZIMUTH (A) DEGREES	CONF INT ELEV METERS
A	A= 0.020	7	38	51	0.018	0.012	52	0.014
	B= 0.018	83	3	6				
	C= 0.014	1	52	38				
B	A= 0.026	77	8	10	0.018	0.014	125	0.020
	B= 0.020	13	33	54				
	C= 0.016	1	55	35				
C	A= 0.019	3	40	50	0.017	0.013	130	0.014
	B= 0.018	86	0	4				
	C= 0.014	3	50	40				
D	A= 0.025	2	21	69	0.022	0.017	110	0.017
	B= 0.023	74	14	8				
	C= 0.018	16	65	19				
E	A= 0.023	28	35	43	0.020	0.013	50	0.016
	B= 0.020	62	16	22				
	C= 0.014	2	51	39				
F	A= 0.016	43	42	16	0.015	0.012	144	0.012
	B= 0.016	42	26	37				
	C= 0.014	16	37	49				
G	A= 0.024	55	22	26	0.020	0.013	131	0.018
	B= 0.021	35	33	38				
	C= 0.015	1	49	41				
H	A= 0.060	0	2	88	0.054	0.019	88	0.006
	B= 0.021	0	88	2				
	C= 0.008	90	0	0				
I	A= 0.144	0	83	7	0.130	0.026	7	0.006
	B= 0.029	0	7	83				
	C= 0.007	90	0	0				

99 % RELATIVE CONFIDENCE ANALYSIS

APRIORI VARIANCE FACTOR = 1.000

EXPANSION FACTORS = [2.576 (1D), 3.035 (2D), 3.368 (3D)].

STA NAME	STA NAME	3 DIMENSIONAL ELLIPSOIDS AXES METERS	ORIENTATION TO RESPECTIVE PLANES DEGREES NE	EH	NH	2D PROJECTED CONFIDENCE ELLIPSES A METERS	B	AZIMUTH DEGREES	CONF INT (A) ELEV METERS
A	B	A= 0.032	79	10	4	0.023	0.021	89	0.024
		B= 0.025	3	3	85				
		C= 0.022	11	79	3				
A	C	A= 0.026	75	7	13	0.022	0.021	88	0.020
		B= 0.025	13	1	77				
		C= 0.023	7	83	0				
A	D	A= 0.032	44	24	36	0.026	0.022	126	0.022
		B= 0.027	45	27	32				
		C= 0.024	3	52	38				
A	E	A= 0.029	53	21	29	0.024	0.018	52	0.021
		B= 0.026	37	30	39				
		C= 0.020	1	52	38				
A	F	A= 0.025	79	10	5	0.021	0.019	44	0.019
		B= 0.024	4	44	45				
		C= 0.021	11	44	44				
A	G	A= 0.030	69	10	18	0.024	0.022	119	0.023
		B= 0.026	21	26	56				
		C= 0.024	1	62	28				
A	H	A= 0.060	6	2	84	0.054	0.024	92	0.015
		B= 0.027	6	84	2				
		C= 0.019	82	6	6				
A	I	A= 0.142	2	83	7	0.128	0.030	7	0.015
		B= 0.033	1	7	83				
		C= 0.019	87	3	1				
B	C	A= 0.032	76	12	8	0.025	0.018	129	0.024
		B= 0.028	13	36	51				
		C= 0.019	4	51	38				
B	D	A= 0.035	81	1	9	0.029	0.022	120	0.026
		B= 0.032	7	30	59				
		C= 0.024	5	60	30				
B	E	A= 0.034	83	4	6	0.024	0.022	73	0.026
		B= 0.027	4	19	71				
		C= 0.024	5	71	18				
B	F	A= 0.031	80	7	6	0.023	0.018	134	0.023
		B= 0.026	10	43	46				
		C= 0.020	1	46	44				

B	G	A= 0.035	87	1	3	0.027	0.018	130	0.027
		B= 0.030	1	41	49				
		C= 0.020	3	49	40				
B	H	A= 0.063	0	0	90	0.057	0.025	90	0.021
		B= 0.028	45	45	0				
		C= 0.026	45	45	0				
B	I	A= 0.143	2	83	7	0.129	0.032	7	0.021
		B= 0.036	16	6	73				
		C= 0.026	74	3	16				
C	D	A= 0.031	9	30	58	0.028	0.021	119	0.022
		B= 0.029	77	3	13				
		C= 0.023	9	60	28				
C	E	A= 0.028	79	11	3	0.023	0.022	64	0.021
		B= 0.026	7	23	66				
		C= 0.024	8	64	24				
C	F	A= 0.025	15	42	44	0.023	0.016	135	0.019
		B= 0.024	75	14	7				
		C= 0.018	4	45	45				
C	G	A= 0.031	47	23	34	0.026	0.016	132	0.023
		B= 0.028	42	34	29				
		C= 0.018	7	47	43				
C	H	A= 0.062	1	1	89	0.056	0.024	91	0.015
		B= 0.027	3	87	0				
		C= 0.020	87	3	1				
C	I	A= 0.144	1	83	7	0.130	0.031	7	0.015
		B= 0.035	5	6	82				
		C= 0.020	85	1	5				
D	E	A= 0.032	55	20	28	0.027	0.023	116	0.024
		B= 0.028	33	10	55				
		C= 0.026	11	68	19				
D	F	A= 0.030	20	32	51	0.027	0.021	120	0.021
		B= 0.028	64	3	26				
		C= 0.023	16	58	27				
D	G	A= 0.034	38	28	39	0.030	0.022	122	0.025
		B= 0.031	51	13	36				
		C= 0.024	8	58	31				
D	H	A= 0.062	11	12	74	0.055	0.024	102	0.018
		B= 0.027	15	73	9				
		C= 0.021	72	12	13				
D	I	A= 0.146	1	84	6	0.132	0.033	6	0.018
		B= 0.037	16	5	73				
		C= 0.022	74	3	16				
E	F	A= 0.027	59	28	12	0.023	0.020	41	0.020
		B= 0.025	28	33	44				
		C= 0.022	13	44	43				

E	G	A= 0.032	70	17	10	0.024	0.023	111	0.024
		B= 0.027	10	2	80				
		C= 0.026	17	73	1				
E	H	A= 0.061	5	1	85	0.055	0.025	91	0.017
		B= 0.028	3	87	2				
		C= 0.022	84	3	5				
E	I	A= 0.143	1	83	7	0.129	0.031	7	0.017
		B= 0.034	4	7	82				
		C= 0.022	86	2	4				
F	G	A= 0.029	52	23	28	0.025	0.017	135	0.021
		B= 0.026	37	36	32				
		C= 0.019	4	45	45				
F	H	A= 0.062	0	1	89	0.056	0.024	89	0.014
		B= 0.026	3	87	1				
		C= 0.018	87	3	0				
F	I	A= 0.144	0	83	7	0.130	0.030	7	0.013
		B= 0.034	5	7	81				
		C= 0.017	85	1	5				
G	H	A= 0.063	2	1	88	0.057	0.025	91	0.019
		B= 0.028	9	81	2				
		C= 0.024	81	9	1				
G	I	A= 0.145	0	84	6	0.130	0.032	6	0.019
		B= 0.036	1	6	84				
		C= 0.024	89	0	1				
H	I	A= 0.145	1	84	6	0.130	0.057	6	0.005
		B= 0.064	7	6	80				
		C= 0.005	83	0	7				

DIAGONAL ELEMENTS OF COVARIANCE MATRIX (Cx)

(1, 1) = 0.281E-04
(2, 2) = 0.238E-04
(3, 3) = 0.300E-04
(4, 4) = 0.311E-04
(5, 5) = 0.261E-04
(6, 6) = 0.597E-04
(7, 7) = 0.262E-04
(8, 8) = 0.235E-04
(9, 9) = 0.297E-04
(10,10) = 0.517E-04
(11,11) = 0.331E-04
(12,12) = 0.453E-04
(13,13) = 0.330E-04
(14,14) = 0.281E-04
(15,15) = 0.387E-04
(16,16) = 0.188E-04
(17,17) = 0.208E-04
(18,18) = 0.225E-04
(19,19) = 0.327E-04

```
(20,20) = 0.294E-04
(21,21) = 0.468E-04
(22,22) = 0.320E-03
(23,23) = 0.406E-04
(24,24) = 0.598E-05
(25,25) = 0.982E-04
(26,26) = 0.180E-02
(27,27) = 0.487E-05
```